THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and ninety-sixth regular meeting of the Society was held in New York City on Saturday, February 23, extending through the usual morning and afternoon sessions. The attendance included the following seventeen members:

Dr. F. W. Beal, Professor F. N. Cole, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. R. Hollcroft, Dr. L. L. Jackson, Mr. S. A. Joffe, Dr. J. R. Kline, Professor E. J. Miles, Professor R. L. Moore, Mr. George Paaswell, Dr. J. F. Ritt, Dr. Caroline E. Seely, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams.

Professor H. S. White occupied the chair. The Council announced the election of the following persons to membership in the Society: Miss M. F. Chadbourne, Smith College; Mr. Mervyn Davis, Equitable Life Insurance Company of Iowa; Mr. T. C. Fry, Western Electric Company; Dr. J. E. McAtee, University of Illinois; Dr. Norbert Wiener, Albany, N. Y. Four applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Dr. J. F. RITT: "Proof of the multiplication formula for determinants by means of linear differential equations."

(2) Dr. OLIVE C. HAZLETT: "On vector covariants."

(3) Dr. P. R. RIDER: "On the problem of the calculus of variations in n dimensions."

(4) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group."

(5) Dr. A. R. SCHWEITZER: "On certain articles on functional equations."

(6) Dr. A. R. SCHWEITZER: "On iterative functional equations of the distributive type."

(7) Dr. J. R. KLINE: "A new proof of a theorem due to Schoenflies."

(8) Professor R. L. MOORE: "A sufficient condition that a system of arcs should constitute a surface."

(9) Mr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(10) Professor O. E. GLENN: "Covariant expansion of a modular form."

(11) Dr. J. F. RITT: "Polynomials with a common iterate."

(12) Professor L. P. EISENHART: "Transformations of applicable conjugate nets of curves on surfaces."

(13) Professor S. E. SLOCUM: "The romantic aspect of numbers."

Mr. Walsh's paper was communicated to the Society through Professor Bôcher. In the absence of the authors the papers of Dr. Hazlett, Dr. Rider, Dr. Schweitzer, Mr. Walsh, Professor Glenn, Professor Eisenhart, and Professor Slocum were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Ritt's paper, which is presented only as an interesting piece of manipulation, shows how the formula for the multiplication of determinants can be found from the properties of Wronskian determinants.

2. This paper concerns itself with a certain type of covariant of the general linear algebra which, for convenience, is called a vector covariant. In a previous paper, Dr. Hazlett has developed the theory of a rather different type of covariant, referred to here as scalar covariants, since they involve only the coordinates x_i of the general number of the algebra $X = \Sigma x_i e_i$ and the constants of multiplication γ_{ijk} $(i, j, k = 1, \ldots, n)$ in contradistinction to the vector covariants which involve the units e_i $(i = 1, \ldots, n)$ in addition to the x's and γ's. Since multiplication of the units in general is neither commutative nor associative, one might anticipate some rather inconvenient difficulties. These can, however, be avoided by a device. Accordingly, the annihilators of vector covariants are found and hence it is shown that every rational integral vector covariant of the n-ary linear algebra is a covariant of the general number $X = \Sigma x_i e_i$ of the algebra and a suitable set of scalar covariants of the algebra. From this fact flow theorems analogous to those proved for scalar covariants and and, in particular, the "finiteness" of vector covariants.

3. Dr. Rider's paper considers the theory of minimizing an

integral of the form

$$\int_{t_0}^{t_1} f(x_1, \cdots, x_n, \tau_1, \cdots, \tau_{n-1}) \sqrt{x_1'^2 + \cdots + x_n'^2}\, dt,$$

in which

$$\tau_i = \arctan x_{i+1}'/\sqrt{x_1'^2 + \cdots + x_i'^2}.$$

The Euler equations are derived and the transversality and corner conditions are given, as are the forms of the Hilbert invariant integral and the Weierstrass e-function. Certain necessary conditions, including the condition of Weierstrass, are obtained. The paper will be published in the *Tôhoku Mathematical Journal*.

4. Dr. Schweitzer gives the following postulates* for an Abelian group: I_1. x, y imply $\phi(x, y) = z$ uniquely. I_2. x, y imply $f(x, y) = z$ uniquely. II_1. $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$. II_2. $\phi\{f(x, y), x\} = x$. II_3. $f\{\phi(x, y), x\} = x$. II_4. $\phi(x, y) = \phi(y, x)$ if $x \neq y$. The author shows that if II_4 is replaced by II_4', $\phi(y, y') = \phi(y, y)$ implies $y = y'$ if $y \neq y'$, then the postulates are satisfied by any group, finite or infinite. In the latter set postulate II_1 may be replaced by II_1', $f\{f(y, x), f(z, x)\} = f(y, x)$, without loss of generality.

In the second part of the paper, on the basis of the preceding postulates, the author points out concepts in group theory expressible in terms of iterative compositions, such as the transform of an element, the commutator of two elements. In particular, infinite sequences of iterative compositional relations are obtained which are satisfied by the elements of any group. Such relations are, e. g., of the type $\psi\{u_1^{(i)}, u_2^{(i)}, \cdots, u_n^{(i)}\} = \psi\{x_1, x_2, \cdots, x_n\}$ where $u_i^{(1)} = \psi\{x_i, t_1, t_2, \cdots, t_{n-1}\}$, $u_i^{(2)} = \psi\{t_1, x_i, t_2, \cdots, t_{n-1}\}$ etc., and $i = 1, 2, \cdots, n$ $(n = 2, 3, \cdots)$.

The preceding postulates are free from explicit statements of existence. In the *Transactions* (1905) Dickson in effect eliminates the explicit existential properties of the Moore-Dickson postulates for a finite group.

5. Dr. Schweitzer's second paper is a constructive criticism of certain articles† on functional equations due to Suto on the

* Cf. Grassmann, Gesammelte Werke, vol. I_2, p. 239.
† *Tôhoku Math. Journal*, vol. 3 (1913), vol. 6 (1914–1915).

basis of the author's independent investigations. Numerous errors of Suto are pointed out; in particular the methods by which the iterative functional equation, loc. cit., volume 3, page 60, is solved are erroneous. This equation is but one of an important class of iterative equations previously defined by the author, e. g.,

$$f\{u_1, u_2, \cdots, u_n\} = f\{v_1, v_2, \cdots, v_n\} \quad (n = 2, 3, \cdots),$$

where

$$u_i = f\{x_{1i}, x_{2i}, \cdots, x_{ni}\}, \; v_i = f\{x_{j_1k_1}, x_{j_2k_2}, \cdots, x_{j_nk_n}\}$$

and $x_{j_1k_1}$ is some one of the arguments of u_1, $x_{j_2k_2}$ is some one of the arguments of u_2, etc., and $i = 1, 2, \cdots, n$. The equation treated by Suto was solved by the author by direct differentiation. Concerning the "associative" functional equations the author obtains a number of theorems the simplest of which is: If $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$ and $\phi(x', y) = z$ implies $x' = f(z, y)$ then $\phi(x, y) = \psi^{-1}\{\psi(x) + \psi(y)\}$ and $f(x, y) = \psi^{-1}\{\psi(x) - \psi(y)\}$. The equation of Suto, loc. cit., volume 6, page 82, is a special case of the equation

$$\sum_{i=1}^{n} f_i(x, y) = \psi(x, y),$$

where the functions $f_i(x, y)$ satisfy given partial differential equations of the first order and $\psi(x, y)$ is a given "quasi-transitive" or "symmetric" function, e. g., $x - y$, x/y, $x + y$, $x \cdot y$. In particular important functional equations arise if the $f_i(x, y)$ are required to satisfy given iterative functional equations. Finally Suto's paper on the arithmetic mean, loc. cit., volume 6 (page 79), is compared with the author's.

6. Dr. Schweitzer proves the following theorem: A sufficient condition that

$$\begin{aligned} &\phi\{\phi(x_{11}, \cdots, x_{1n}), t_1, t_2, \cdots, t_{n-1}\} \\ &\quad = \phi\{\phi(x_{11}, t_1, \cdots, t_{n-1}), \cdots, \phi(x_{1n}, t_1, \cdots, t_{n-1})\}, \end{aligned} \tag{1}$$

where $n = 2, 3, \cdots$, is that

$$\begin{aligned} &\phi\{\phi(x_{11}, \cdots, x_{1n}), \cdots, \phi(x_{n1}, \cdots, x_{nn})\} \\ &\quad = \phi\{\phi(x_{11}, \cdots, x_{n1}), \cdots, \phi(x_{1n}, \cdots, x_{nn})\} \end{aligned} \tag{2}$$

and

$$\phi\{\phi(x_{11}, \cdots, x_{1n}), \phi(t_1, \cdots, t_1), \cdots, \phi(t_{n-1}, \cdots, t_{n-1})\}$$
$$= \phi\{\phi(x_{11}, \cdots, x_{1n}), t_1, t_2, \cdots, t_{n-1}\}$$

and consequently infers a solutions of (1). The result is generalized to iterative equations of infinite degree.* The equation (1) is a distributive equation of order 1; equation (2) is a limiting case, viz., of order n. Equations of order k, $1 \leqq k \leqq n$ are readily defined, e. g.,

$$\phi\{\phi(x_1, x_2, x_3)\phi(y_1, y_2, y_3)t\} = \phi\{\phi(x_1, y_1, t)\phi(x_2, x_2, t)\phi(x_3, y_3, t)\}$$

is distributive of the second order. Extension to distributive iterative equations on two functions is made.

7. In a paper, presented to the Society at the December meeting, Professor R. L. Moore proved the following theorem: If the points A and C separate B and D on the simple closed curve J, then it is possible to draw a set of simple continuous arcs such that through every point of J and its interior there are two and only two arcs of the set, the arcs AB, BC, CD and DA being arcs of the set. This theorem was proved on the basis of his set of axioms Σ_1†. With the use of this theorem Dr. Kline gives a simple proof of the following theorem, due to Schoenflies: Given two simple closed plane curves J_1 and J_2, which are in one-to-one continuous correspondence under a correspondence Π. Let R_i $(i = 1, 2)$ denote the point set composed of J_i plus I_i, the interior of J_i. Then there exists a continuous one-to-one correspondence between the points of R_1 and R_2 such that the points of J_1 and J_2 correspond as fixed by the correspondence Π. For an abstract of a proof of this theorem by Dr. G. A. Pfeiffer from a somewhat different point of view see this BULLETIN, volume 24, number 4, page 171.

8. Suppose that AB and CD are two arcs in three-dimensional space. Suppose that G is a bounded self-compact system of simple continuous arcs such that each arc of G has its end points on AB and CD respectively and no two arcs of

* Cf. Suto, loc. cit., vol. 6 (1914–1915), p. 16.

† Cf. R. L. Moore, "On the foundations of plane analysis situs," *Transactions Amer. Math. Society*, volume 17 (1916), pp. 131–164.

G have any point in common. Professor Moore proposes to show: (1) that the point set constituted by such a system of arcs cannot always be brought into continuous one-to-one correspondence with the surface consisting of a circle plus its interior; (2) that such a correspondence will exist in case the arcs of the system G are equicontinuous. He has under consideration further theorems of this type.

9. Mr. Walsh's paper will be published in the *Transactions*.

10. Professor Glenn's paper contains elaboration, and computations based on particular cases, of a former theorem of his concerning the reducibility (mod p) of the general binary quantic. The forms of orders $> p^2-1$ ($p = 2, 3$) are reducible modulo p in terms of modular covariants of orders 0 to p^2-1, as polynomials or covariant expansions in the universal covariants. Such an expansion is a form of generalized modular typical representation in which the multiplier concomitant is unity. Tables are given showing the expansions and all constituent covariants occurring in them, for forms of the first eleven orders modulo 2, and for the orders 9, 10, 11 when the modulus is 3.

The paper appeared in the March, 1918, number of the *Annals of Mathematics*.

11. Dr. Ritt determines the conditions under which two different polynomials $F(x)$ and $f(x)$, both of degree n greater than one, can have a common pth iterate. $F(x)$ can be derived by means of a linear integral transformation from a polynomial $g(x) = x^n + k_1x^{n-q} + k_2x^{n-2q} + \cdots k_rx^{n-rq}$, where q is some divisor of $(n^p - 1)/(n - 1)$, and $f(x)$ will be derived by the same transformation from $\omega g(x)$, where ω is a qth root of unity. The proof is accomplished with the help of an important theorem on iteration due to L. Boettcher, which has lain hidden for a long time in a paper published in Russian.*

12. When two surfaces are applicable to one another, in the correspondence thus established there is a unique conjugate system of curves on each surface corresponding to a conjugate system on the other. These two conjugate systems, or nets, are said to be applicable nets. Let N and $\overline{N}$ be two such applicable nets. Peterson showed that when a net N', parallel

* *Bulletin of the Kasan Math. Society*, vol. 14 (1905) p. 176.

to N, is known, a net $\overline{N}'$, parallel to $\overline{N}$, can be found by quadratures such that N' and $\overline{N}'$ are applicable. Professor Eisenhart has applied to applicable nets the theory of transformations T (cf. *Transactions*, volume 17, page 97), and has established transformations of applicable nets into applicable nets. The transformations of nets admitting an infinity of applicable nets into similar nets are of particular interest. The paper will be published in the April number of the *Transactions*.

13. Professor Slocum's paper calls attention to the important rôle played by number symbolism in the intellectual development of mankind. Wherever mind has reacted to the stimulus of natural phenomena, the number concept has resulted as the inevitable expression of the laws governing the material universe. The properties of number, first accepted as a fact, subsequently came to be regarded as symbolic, and only in recent times has the mind been able to grasp their true significance as one aspect of the great principle of functionality.

Various instances are cited showing that the ancient customs of all races imply a distinct belief in the action of numbers on the course of human events. Beginning with the empiricism of primitive nations, like the Egyptians, Persians, and Greeks, it is shown that as civilization developed number ideas became symbolic, and the properties of numbers were invested with physical attributes. Numerals were regarded not merely as passive symbols but as active principles of good and evil, capable of reproducing their properties in accordance with mathematical operations. The extensive use of number symbolism by the ancient Hebrews, however, stands in marked contrast to that characteristic of pagan nations like the Greeks. With the Hebrews number symbolism was simply incidental to their rich Oriental imagery, and there is no evidence anywhere in Old Testament literature that the properties of numbers were even known, much less regarded as exerting any mystic influence on human destiny. But when the New Testament was written the Hebrews had outgrown their isolation, and Grecian influence is clearly apparent in the mystic character given to number symbolism, especially in the Apocalypse.

During the middle ages the attempt of scholasticism to reconcile the tenets of Christianity with Grecian philosophy

lent special impetus to the study and development of sacred hermeneutics. The growth of this peculiar by-product of scholasticism is outlined, with references to the work on number symbolism of the Venerable Bede, St. Augustine, and numerous other prominent ecclesiastics. In particular, a remarkable example is given of a ninth century rendering of the account given in Genesis 18 of the conflict of the 318 servants of Abraham against the four kings. The classic example of the number of the beast is also studied, and it is shown that its interpretation serves to establish the date of the writing of Revelation, long in dispute, as well as affords a key to the system of number symbolism used by St. John.

F. N. COLE,
Secretary.

SOME REMARKABLE DETERMINANTS OF INTEGERS.

BY PROFESSOR E. T. BELL.

1. THE determinants in this note are arithmetical rather than algebraic in character; their properties, not obvious by the usual reductions, follow immediately from simple considerations in the theory of numbers. Throughout, letters other than x and functional signs denote positive integers, and $[x]$ is the greatest integer in x.

2. Let $D_n\,|F(k),\ G(k)\,|$ denote the determinant of the nth order whose first and last columns are respectively $F(1)$, $F(2)$, $\cdots$, $F(n)$, and $G(1)$, $G(2)$, $\cdots$, $G(n)$; and whose $(1+k)$th column ($k = 1, 2, \cdots, n-2$) is derived from the first by prefixing k zeros and repeating in succession each element of the first $(1+k)$ times, until in all a column of n elements has been written down

$$(1)\qquad D_n\,|F(k),\ G(k)\,| \equiv \begin{vmatrix} F(1) & 0 & 0 & \cdots & G(1) \\ F(2) & F(1) & 0 & \cdots & G(2) \\ F(3) & F(1) & F(1) & \cdots & G(3) \\ F(4) & F(2) & F(1) & \cdots & G(4) \\ F(5) & F(2) & F(1) & \cdots & G(5) \\ F(6) & F(3) & F(2) & \cdots & G(6) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{vmatrix}.$$

For a reason appearing presently, $D_n|G(k), F(k)|$ is called the inverse of (1). A genesis of (1) is as follows: Let

$$(2) \qquad G(n) = \sum_{a=1}^{n} g(a)F([n/a]); \quad F(1) = 1; \quad F(0) = 0.$$

Then, putting in (2) successively $n = 1, 2, \cdots, n$, and solving the resulting system for $g(n)$, we find $g(n) = D_n|F(k), G(k)|$.

3. The more interesting determinants (1) relate to $F \equiv f'$, $G \equiv \overline{gf'}$, where the notation is

$$(3) \quad f'(n) \equiv \sum_{a=1}^{n} f(a); \quad \overline{gf}(n) \equiv \sum_{n} g(d)f(n/d); \quad \overline{gf}'(n) \equiv \sum_{a=1}^{n} \overline{gf}(a),$$

the $\sum_{n}$ extending to all divisors $1, d, \cdots, n$, of n. Clearly, $\overline{gf}(n) = \overline{fg}(n)$, and $\overline{gf}'(n)$ denotes $\Sigma g(d)f(\delta)$, the $'\Sigma$ referring to all d, δ such that $d\delta \leqq n$; whence, noting the number of times that d takes the particular value a, obviously

$$(4) \qquad \overline{gf}'(n) = \sum_{a=1}^{n} g(a)f'([n/a]) = \sum_{a=1}^{n} f(a)g'([n/a]).$$

Comparing (4), (2), we have, for $f(1) = g(1) = 1$, and therefore $f'(1) = g'(1) = 1$

$$(5) \quad g(n) = D_n|f'(k), \overline{gf}'(k)|; \; f(n) = D_n|g'(k), \overline{gf}'(k)|.$$

4. To limit the paper, henceforth g, f denote ψ-functions, viz., solutions of $\psi(1) = 1$, $\psi(mn) = \psi(m)\psi(n)$ for m, n relatively prime. It may be shown without difficulty that when g is given, there always exists a unique ψ-function, g_1, such that $\overline{gg}_1(1) = 1$, and $\overline{gg}_1(n) = 0$ for $n > 1$. We shall call such g, g_1 reciprocals of each other; and it is easily verified that for g, g_1 and f, f_1 reciprocals, (4) implies the inversions

$$(6) \quad f'(n) = \sum_{a=1}^{n} g_1(a)\overline{gf}'([n/a]); \; g'(n) = \sum_{a=1}^{n} f_1(a)\overline{gf}'([n/a]);$$

whence in all, as consequences of (4), the pairs of inverses

$$(7) \quad \left.\begin{aligned} f(n) &= D_n|g'(k), \overline{gf}'(k)| \\ f_1(n) &= D_n|\overline{gf}'(k), g'(k)| \end{aligned}\right\}; \quad \left.\begin{aligned} g(n) &= D_n|f'(k), \overline{gf}'(k)| \\ g_1(n) &= D_n|\overline{gf}'(k), f'(k)| \end{aligned}\right\}.$$

Again, from (7) and the definitions directly, if $\overline{gf}(n) = h(n)$

$$(8) \qquad h(n) = \sum_{n} \{D_d \,|\, g'(k), \overline{gf}'(k) \,|\cdot D_{n/d} \,|\, f'(k), \overline{gf}'(k) \,|\};$$

and, noting that f, f_1 and g, g_1 are reciprocals, we have for $n > 1$

$$(9) \qquad 0 = \sum_{n} \{D_d \,|\, g'(k), \overline{gf}'(k) \,|\cdot D_{n/d} \,|\, \overline{gf}'(k), g'(k) \,|\},$$

and the like with g, f interchanged. Also, for m, n relatively prime (but not otherwise)

$$(10) \quad D_{mn} \,|\, f'(k), \overline{gf}'(k) \,| = D_m \,|\, f'(k), \overline{gf}'(k) \,|\cdot D_n \,|\, f'(k), \overline{gf}'(k) \,|,$$

and similarly with g, f interchanged. Hence, if $n = p^a \cdots r^b$ is the resolution of n into prime factors, $D_n \,|\, f'(k), \overline{gf}'(k) \,|$ = the product of similar determinants of respective orders $p^a, \cdots, r^b$.

5. At least one pair of the (7) will reduce to determinants of pure integers, viz., free from all functional signs f', g', $\overline{gf}'$, for such f, g as give one of $f'(k)$, $g'(k)$, and $\overline{gf}'(k)$ explicit functions of k. In the theory of numbers there are a great many such f, g: to illustrate § 4 we select in § 6 a few of the simplest. Again, for f, g given ψ-functions, it is known, or may be proved easily from first principles, that there always exists a unique ψ-function h, in general simpler than $\overline{fg}$, such that $\overline{fg}(n) = h(n)$. For specific f, g this reduction is readily effected when $n = p^a$, p prime, and hence immediately from the definition of ψ-functions for general n; similarly for the reciprocal f_1 of specific f. All the like reductions quoted in § 6 are either well known or may be thus verified with ease. We shall require the definition: a number which is divisible by no square > 1 is *simple*.

6. In illustration, consider the determinants dependent upon the following set of intimately related ψ-functions and their simpler properties: (i) $u_r(n) \equiv n^r$, whose reciprocal for $r > 0$, is $u_r(n)\mu(n)$. Also, u_0, μ are reciprocals; hence $\overline{u_0\mu}'(k) = 1$. Clearly, $u_0'(k) = k$; $u_1'(k) = \frac{1}{2}k(k+1)$. (ii) $\varpi(n) \equiv (-1)^{\lambda(n)}$, $\lambda(n)$ the *total* number of prime divisors of n. Writing $\mu(n) \times \mu(n) \equiv \mu^2(n)$; ϖ, μ^2 are reciprocals, whence

$$\overline{\varpi\mu^2}'(k) = 1.$$

Also $\overline{\mu^2 t_2} = u_0$, $\overline{u_0 \varpi} = t_2$; whence $\overline{u_0 \varpi}'(k) = t_2'(k) = [\sqrt{k}]$; $\overline{\mu^2 t_2}'(k) = u_0'(k) = k$. (iii) $\mu(n) = 0$ or $\varpi(n)$ according as n is not, or is, simple (Möbius' function). (iv) $t_r(n) = 1$ or 0 according as n is, or is not, an rth power > 1. The reciprocal is $t_r(n)\mu(\sqrt[r]{n})$; and $t_r'(k) = [\sqrt[r]{k}]$.

We apply the $f(n)$, $f_1(n)$ pair in (7) to these. For $f \equiv \mu$, $g \equiv u_0$, by (i)

$$\mu(n) = D_n|k, 1|; \quad 1 = D_n|1, k|. \tag{11}$$

For $f \equiv \mu^2$, $g \equiv t_2$, (ii) gives

$$\mu^2(n) = D_n|[\sqrt{k}], k|; \quad \varpi(n) = D_n|k, [\sqrt{k}]|. \tag{12}$$

From the reciprocals (by (i)), $f(n) \equiv n\mu(n)$, $g(n) \equiv u_1(n)$,

$$n\mu(n) = D_n|\tfrac{1}{2}k(k+1), 1|; \quad n = D_n|1, \tfrac{1}{2}k(k+1)|. \tag{13}$$

To complete this set we add $\varphi_r(n)$ and Gegenbauer's $\mu_r(n)$; (v) $\mu_r(n) = 0$ if n is divisible by an rth power ($r > 1$), otherwise $= 1$. The reciprocal, γ_r, is defined by $\overline{\mu t_r} = \gamma_r$. Also, $\overline{\mu_r t_r} = u_0$; $\mu_2 = \mu^2$. (vi) $\varphi_r(n)$ = the number of integers $\leqq n^r$ that are divisible by the rth power of no prime divisor of n^r; $\varphi_1 \equiv \varphi$, the ordinary totient. The reciprocal, χ_r, is defined by $\chi_r(p^a) = (1 - p^r)$; also $\overline{u_0 \varphi_r} = u_r$.

Whence, similarly to (11)–(13), we have for $f \equiv u_r$, $g \equiv t_r$; and for $f \equiv \varphi_r$, $g \equiv u_0$,

$$\mu_r(n) = D_n|[\sqrt[r]{k}], k|; \quad \gamma_r(n) = D_n|k, [\sqrt[r]{k}]|. \tag{14}$$

$$\begin{aligned} \varphi_r(n) &= D_n|k, (1^r + 2^r + \cdots + k^r)|; \\ \chi_r(n) &= D_n|(1^r + 2^r + \cdots + k^r), k|. \end{aligned} \tag{15}$$

For $r = 2$, (14) becomes (12); and for $r = 1$, writing $\chi_1 \equiv \chi$, (15) gives

$$\varphi(n) = D_n|k, \tfrac{1}{2}k(k+1)|; \quad \chi(n) = D_n|\tfrac{1}{2}k(k+1), k|. \tag{16}$$

The theory of numbers furnishes a ready means of writing down any desired quantity of such results. Enough have been given for the purpose of showing how, combined with the definitions of specific ψ's, we may, as in § 7, deduce properties of the determinants themselves.

7. One class of relations between the D_n is obtained on

substituting in (8), (9), (10). E. g., by (10), (11) for m, n relatively prime: $D_{mn}|k, 1| = D_m|k, 1|\cdot D_n|k, 1|$; etc. As examples of another, we have obviously $\mu^{2s+1}(n) = \mu(n)$, $\mu^{2r}(n) = \mu^2(n)$; whence from (11), (12)

$$(17)\qquad \begin{aligned} D_n{}^r|[\sqrt[r]{k}], k| &= D_n|[\sqrt[r]{k}], k| = D_n{}^2|k, 1|; \\ D_n{}^{2s+1}|k, 1| &= D_n|k, 1|. \end{aligned}$$

Again, from (11), (13)

$$(18)\qquad D_n|1, \tfrac{1}{2}k(k+1)|\cdot D_n|k, 1| = D_n|\tfrac{1}{2}k(k+1), 1|.$$

These suffice for this kind. Curious results are found when n is prime; we illustrate, however, another kind of relation which gives less obvious properties of the D_n. Let the order n be a simple number s (defined, § 5). Then, it follows readily from the definitions of the functions that $\gamma_r(s) = \varpi(s) = \mu(s)$; $\mu_r(s) = 1$, and $\varphi_r(s) = \varpi(s)\chi_r(s)$. Hence, from (14), (12), (11)

$$(19)\qquad D_s|k, [\sqrt[r]{k}]| = D_s|k, [\sqrt[r]{k}]| = D_s|k, 1|,$$

$$(20)\qquad D_s|[\sqrt[r]{k}], k| = 1 = D_s|1, k|;$$

and from (15), (12)

$$(21)\qquad \begin{aligned} &D_s|k, (1^r + 2^r + \cdots + k^r)| \\ &\qquad = D_s|k, [\sqrt[r]{k}]|\cdot D_s|(1^r + 2^r + \cdots + k^r), k|, \end{aligned}$$

which, combined with (19), (20), gives further s-identities. A totally distinct kind of result may be found by using the g, g_1 pair of (7) instead of, as above, the f, f_1 pair; the D_n in this case contain ψ-functions of k. Last, isomorphic to the ψ-functions there is a class of Ψ-functions having the same formal properties as the ψ's, but based upon the resolution of n into distinct (viz., relatively prime) simple numbers, $P_1, \cdots, P_r$ (thus, $n = P_1{}^a\cdots P_r{}^b$), instead of into distinct primes; and for these Ψ's there are theorems on determinants corresponding to those for the ψ's, including the kind (19) to (21), the correspondents in the latter case being for $S = P_1\cdots P_r$.

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A THEOREM ON SEMI-CONTINUOUS FUNCTIONS.

BY PROFESSOR HENRY BLUMBERG.

(Read before the American Mathematical Society December 1, 1917.)

RECENTLY G. C. Young* and A. Denjoy† have communicated theorems—those in Denjoy's memoir are of an especially comprehensive character—dealing, in particular, with point sets where the four derivatives of a given continuous function are identical. It is the purpose of this note to treat an analogous problem that arises when "derivative" is replaced by "saltus."‡ However, instead of confining ourselves to "saltus," we prove a more general theorem that applies essentially to all semi-continuous functions.§ We preface the proof of this theorem with the following

LEMMA. *Let $f_1(x)$ and $f_2(x)$ be two real, single-valued functions, defined in the linear continuum, such that everywhere $f_1(x) \geqq f_2(x)$, and moreover, for every fixed real number k, the set S_k of points x where $f_1(x) \geqq k$ and $f_2(x) < k$ is countable. Then $f_1(x)$ and $f_2(x)$ are identical except at most in a countable set.*

Proof. Let $\{k_n\}$, $n = 1, 2, \cdots \infty$, be a set of k's everywhere dense in the linear continuum. The set|| $S = \mathfrak{S}(S_{k_n})$, which consists of all of the elements of every S_{k_n}, is also countable. We show that $f_1(x) = f_2(x)$ for every given x not in S. For let $\{k_{i_n}\}$ be a monotone decreasing sequence of k_n's having $f_2(x)$ as limit. Since x is given as not belonging to S, it must be that $f_1(x) < k_{i_n}$ for every n; for from $f_1(x) \geqq k_{i_n}$ and $f_2(x) < k_{i_n}$, we would conclude that x belonged to $S_{k_{i_n}}$ and

* *Acta Mathematica*, vol. 37 (1914), p. 141.

† *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 105.

‡ By the saltus (= oscillation) of a given function $f(x)$ at the point x we understand the greatest lower bound of the saltus of $f(x)$ in the interval (ξ, η), for all intervals (ξ, η) enclosing x as interior point. Cf. Hobson, The Theory of Functions of a Real Variable (1907), art. 180, and the author's paper, "Certain general properties of functions," *Annals of Mathematics*, vol. 18 (1917), p. 147. For the comprehensiveness of our result see the remark at the close, in conjunction with the theorem of this note and the corollaries.

§ "Essentially" in the sense that every semi-continuous function is exhibitable as the "associated" function $\phi(x)$ of a "monotone decreasing interval-function ϕ_{ab}" (see theorem below).

|| In regard to the notation, compare Hausdorff, Grundzüge der Mengenlehre (1914), p. 5.

hence to S. It follows that $f_1(x) \leqq f_2(x)$, and hence, since $f_1(x) \geqq f_2(x)$ by hypothesis, that $f_1(x) = f_2(x)$.

Definitions. ϕ_{ab} is said to be a "monotone, decreasing, real interval function,"* if ϕ_{ab} is a real number for every (closed) interval (a, b) and, in addition, $\phi_{\alpha\beta} \leqq \phi_{ab}$ if (α, β) is contained in (a, b). With every such interval function, we associate three point functions: (1) $\phi(x)$, the greatest lower bound of all ϕ_{ab} such that the fixed point x is in the interior of (a, b); (2) $\phi^{(r)}(x)$, the greatest lower bound of ϕ_{xb}, for all b's greater than x; and (3) $\phi^{(l)}(x)$, the greatest lower bound of ϕ_{ax} for all a's less than x.

The functions $\phi(x)$, $\phi^{(r)}(x)$ and $\phi^{(l)}(x)$ associated with a monotone, decreasing, real interval function ϕ_{ab} that has a finite lower bound necessarily exist, and $\phi(x)$ is an upper semi-continuous function. (For the proof, the reader is referred to a former paper by the author.)†

We now come to the

THEOREM. *Let ϕ_{ab} be a monotone, decreasing, real interval function having a finite lower bound. Then the three associated point functions $\phi(x)$, $\phi^{(r)}(x)$ and $\phi^{(l)}(x)$ (exist and) are identical except at most in a countable set.*

Proof. It is evidently sufficient to show that $\phi(x)$ and $\phi^{(r)}(x)$ are identical except in a countable set. From the definition of these functions, it follows that $\phi(x) \geqq \phi^{(r)}(x)$ for every x. For if (a, b) contains x as interior point, we have, according to the monotoneity property of ϕ, the inequality $\phi_{xb} \leqq \phi_{ab}$. We may thus associate with every number ϕ_{ab} of the set of which $\phi(x)$ is the greatest lower bound a smaller or equal number ϕ_{xb} of the set of which $\phi^{(r)}(x)$ is the greatest lower bound. Let now S_k represent, for a fixed real number k, the set of points x where simultaneously $\phi(x) \geqq k$ and $\phi^{(r)}(x) < k$.‡ If x belongs to S_k, there must exist, on account of the relation $\phi^{(r)}(x) < k$, an interval (x, b) such that $\phi_{xb} < k$. From the last inequality, it follows that $\phi(\xi) < k$ for every ξ interior to (x, b); hence ξ does not belong to S_k. We have thus associated with every point x of S_k an interval (x, b) having no interior points belonging to S_k.§ Clearly two such

* Defined, as is the case with all the functions treated in this note, in the entire linear continuum.

† *Annals of Mathematics*, l. c.

‡ Cf. G. C. Young, l. c., p. 143.

§ This association may be made unique by taking the point b as far to the right as possible, and allowing, if necessary, the "ideal" interval (x, ∞).

associated intervals cannot overlap. Since a set of non-overlapping intervals in the linear continuum is at most countable, it follows that S_k is at most countable. By the use of the lemma we now conclude, identifying $f_1(x)$ with $\phi(x)$ and $f_2(x)$ with $\phi^{(r)}(x)$, that $\phi(x)$ and $\phi^{(r)}(x)$ differ at most in a countable set.

If, in particular, ϕ_{ab} means the saltus of a given real function $f(x)$ in the interval (a, b), we obtain as associated point functions of ϕ the ordinary saltus and two other functions, which may be designated as the "right saltus" and the "left saltus." We thus have the

COROLLARY. *The three functions $\phi(x)$, $\phi^{(r)}(x)$, and $\phi^{(l)}(x)$, representing respectively the saltus, the right saltus and the left saltus of a given function $f(x)$ at the point x, are identical except at most in a countable set.*

The *f, d, e, n.d.* and *z*-saltus functions, which are the saltus functions when respectively finite, denumerable, exhaustible, non-dense sets and sets of zero measure may be neglected,* likewise yield particular cases of our theorem, and we have the

COROLLARY. *The f, d, e, n.d. and z-saltus functions are identical with the corresponding right and the left saltus functions except at most in a countable set.*

As a special case of our first corollary we have the

COROLLARY. *The points where a given function $f(x)$ is discontinuous but continuous on the right (left) constitute at most a countable set.*

Remark. The exhaustive character of our theorem may be judged from the fact that if $C = \{x_1, x_2, \cdots\}$ is any given countable set of real numbers, a function $f(x)$ exists whose saltus, f-saltus, e-saltus, etc., coincides everywhere respectively with its right saltus, right f-saltus, right e-saltus, etc., except precisely at the points of C. In fact,

$$f(x) = \sum_{x_n \leqq x} \frac{1}{2^n}$$

—i. e., the sum of all the fractions $1/2^n$ such that $x_n \leqq x$—is such a function. For it may be readily shown that $f(x)$ has its right saltus, right *f*-saltus, etc., everywhere 0, and its saltus, *f*-saltus, etc., everywhere 0 except at every point x_n of C, where these saltus functions all take the value $1/2^n$.

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* Cf. *Annals of Mathematics*, l. c.

SURFACES OF REVOLUTION IN THE THEORY OF LAME'S PRODUCTS.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society September 4, 1916.)

IN *Berliner Monatsberichte*, February, 1878, Wangerin discussed the problem of the most general orthogonal surfaces of revolution such that if Laplace's equation be written in coordinates corresponding to these surfaces a solution may be obtained in the form of a Lamé's product with an extraneous factor, i. e.,

$$V = \lambda \cdot R_1 \cdot R_2 \cdot \theta. \tag{1}$$

R_1, R_2, and θ are functions respectively of the parameters of the two families of surfaces and the meridian planes, while λ may involve all three parameters. Wangerin showed that λ is $1/\sqrt{r}$, where r is the distance from the axis of revolution to the point of intersection of the three surfaces. He also deduced certain meridian curves which have been discussed by Haentzschel in Reduction der Potentialgleichung, Berlin, 1893.

In reducing Laplace's equation the terms involving θ are removed in the usual manner, leaving a partial differential equation which, as both writers state, is resolvable into two ordinary differential equations provided F and F_1, its conjugate, can be found which shall satisfy

$$\frac{F'(t+iu)\cdot F_1'(t-iu)}{[F(t+iu)-F_1(t-iu)]^2} = H(t)+K(u). \tag{2}$$

After F is found the two families of meridian curves follow from

$$x+ir = F(t+iu),\ x-ir = F_1(t-iu),\ r = \sqrt{(y^2+z^2)}, \tag{3}$$

by elimination of u and t respectively, thus making t and u the parameters of which R_1 and R_2 are functions as stated above. As will be seen, H and K disappear in the process of finding F, and the principal result in this paper is a direct process of computing H and K, avoiding the method of

substitution in the first member of (2), which must be followed by a separation not always obvious.

Wangerin differentiates (2) successively with respect to t and u respectively, obtaining

$$\begin{aligned}(4)\quad & (F-F_1)[(F-F_1)(F'''F_1'-F'F_1''') \\ & +(F''F_1'-F'F_1'')(F'-F_1') \\ & -2F'F_1'(F''+F_1'')-2(F'+F_1')(F'F_1''+F''F_1')] \\ & -3(F'-F_1')[(F-F_1)(F''F_1'-F'F_1'') \\ & -2F'F_1'(F'+F_1')]=0.\end{aligned}$$

This expression is next differentiated three times with respect to the argument of F and then the derivatives of F_1 are eliminated. From this result comes the final quotation from Wangerin, the following equation *defining* F:

$$(5)\quad \text{“}(F')^2=AF^4+4BF^3+6CF^2+4B'F+A'=R(F).\text{”}$$

Corresponding to (5)

$$(6)\quad (F_1')^2=A_1F_1^4+4B_1F_1^3+6C_1F_1^2+4B_1'F_1'+A_1'.$$

The conjugate imaginary constants in (5) and (6) come from integration but must be taken *real*, because (4) is satisfied by by F, F_1 and their derivatives from (5) and (6) when and only when the following conditions hold:

$$(7)\quad A-A_1=B-B_1=C-C_1=B'-B_1'=A'-A_1'=0.$$

Haentzschel gives as the solution of (5)

$$\begin{aligned}(8)\quad & F(z)=z_0 \\ & +\frac{\sqrt{R(z_0)}\sqrt{S}+\frac{1}{2}R'(z_0)[s-\frac{1}{24}R''(z_0)]+\frac{1}{24}R(z_0)R'''(z_0)}{2[s-\frac{1}{24}R''(z_0)]^2-A\cdot R(z_0)}, \\ & S=4s^3-g_2s-g_3=4(s-\epsilon_1)(s-\epsilon_2)(s-\epsilon_3), \\ (9)\quad & s=\wp(z), \qquad g_2=AA'-4BB'+3C^2, \\ & \qquad\qquad g_3=ACA'+2BCB'-AB'^2-A'B^2-C^3.\end{aligned}$$

The constant z_0 is in general unrestricted. When z_0 is a root

of $R(z) = 0$, (8) reduces to

$$F(z) = z_0 + \frac{\frac{1}{4}R'(z_0)}{s - \frac{1}{24}R''(z_0)}. \tag{10}$$

Formula (8) was published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures.

For $F(t + iu)$ given by (10) and also for

$$F(t + iu) = \wp(t + iu),$$

which is obtainable from (10) by linear transformation, Haentzschel has found (l. c.) the values of H and K in (2) by substitution of the respective values of F and F_1 in the first member. This method evidently requires a considerable knowledge of the properties of F and F_1 in order that the separation of the first member into two functions may be effected, thus obtaining the values of H and K as desired, and an inspection of Haentzschel's presentation will confirm this inference.

This paper gives a method of expressing H and K as functions of F and F_1 and requires only known properties of $\wp(z)$ in addition to the differential equation (5), thus giving explicitly the separation implied in (2). From (2) by differentiation as to t

$$\frac{(F - F_1)(F'F_1'' + F_1'F'') - 2F'F_1'(F' - F_1')}{(F - F_1)^3} = H'(t), \tag{11}$$

in which the arguments of F and F_1 are respectively $t + iu$ and $t - iu$, and it should be noted that F and F_1 are known functions previously determined by (5) and (6) and given by (8).

Although (11) appears to contain functions of u in the first member, the second member shows that it is independent of u and thus true for any value of u, e. g., $(t - c)/i$. This value of u was suggested by the fact that $F_1(t - iu)$ thus becomes $F_1(c)$. If next c is so chosen as to satisfy

$$F_1'(c) = 0, \tag{12}$$

equation (11) will be much simplified. Since F given by (8)

is a function of $\wp$, (12) is satisfied by taking c as ω, the half period of $\wp$, noticing that $\wp'(\omega)$ is 0. Hence (11) becomes

$$\frac{F_1''(\omega)F'(2t-\omega)}{[F(2t-\omega)-F_1(\omega)]^2} = H'(t). \tag{13}$$

Integrating with respect to t gives

$$H(t) = \frac{-F_1''(\omega)}{2[F(2t-\omega)-F_1(\omega)]} + C_1. \tag{14}$$

Similarly

$$K(u) = \frac{F_1''(\omega)}{2[F(2iu+\omega)-F_1(\omega)]} + C_2. \tag{15}$$

The final form of (2) is

$$\frac{F'(t+iu)\cdot F_1'(t-iu)}{[F(t+iu)-F_1(t-iu)]^2} = H(t)+K(u) \tag{16}$$
$$= \frac{F_1''(\omega)}{2}\left[\frac{1}{F(2iu+\omega)-F_1(\omega)} - \frac{1}{F(2t-\omega)-F_1(\omega)}\right].$$

The term $C_1 + C_2$ from the previous equations is omitted as (16) is an identity for $iu = t - \omega$.

Before treating the general value of F by the use of (16) several special cases will be computed. For the simplest case, $F = \wp$, obtainable from (10) by a linear transformation with real coefficients, the following are necessary in applying (16):

$$\wp(\omega) = \epsilon_1,\ \wp'(\omega) = 0,\ \wp''(\omega) = 2(\epsilon_2-\epsilon_1)(\epsilon_3-\epsilon_1),$$

$$\wp(2t-\omega) - \epsilon_1 = \frac{(\epsilon_2-\epsilon_1)(\epsilon_3-\epsilon_1)}{\wp(2t)-\epsilon_1}, \tag{17}$$

$$\wp(2iu+\omega) - \epsilon_1 = \frac{(\epsilon_2-\epsilon_1)(\epsilon_3-\epsilon_1)}{\wp(2iu)-\epsilon_1},$$

whence

$$H(t) + K(u) = \wp(2iu) - \wp(2t). \tag{18}$$

When F is given by (10) and z_0 is a real root of $R(z) = 0$, the result is (18) as in the preceding case, the computation being somewhat longer, though depending only on (17). In this case the coefficients in (10) are real, so that either of the values

of F in the two cases is a real linear transformation of the other. It is a remarkable fact that under such a transformation the fundamental formula (16) is invariant, this being true also of the more general form which (16) would have if c were not specialized.

To verify the proposition above for the last member of (16), let $F(2t-\omega)$ be replaced by $(\alpha F+\beta)/(\gamma F+\delta)$, $\alpha, \beta, \gamma, \delta$ real, and $\alpha\delta-\beta\gamma=\Delta\neq 0$, and similarly for $F(2iu+\omega)$, writing also a like function of W in place of $F_1(\omega)$, noticing that W' will be zero. The result is

$$(19)\quad \frac{W''\Delta}{2(\gamma W+\delta)^2}\left[\frac{(\gamma F_1+\delta)(\gamma W+\delta)}{(F_1-W)\Delta}-\frac{(\gamma F+\delta)(\gamma W+\delta)}{(F-W)\Delta}\right]=\frac{W''}{2}\left[\frac{1}{F_1-W}-\frac{1}{F-W}\right].$$

When at least two of the roots of $R(z)=0$ are imaginary let z_0 be one of the conjugate pair a_1 and a_4; then from (10)

$$(20)\quad F(t+iu)=a_1+\frac{\frac{1}{4}R'(a_1)}{\wp(t-iu)-\frac{R''(a_1)}{24}}=a_1+\frac{M_1}{\wp(t+iu)-N_1},$$

$$F_1(t-iu)=a_4+\frac{\frac{1}{4}R'(a_4)}{\wp(t-iu)-\frac{R''(a_4)}{24}}=a_4+\frac{M_4}{\wp(t-iu)-N_4},$$

$$F_1(\omega)=a_4+M_4/(\epsilon_1-N_4),\ F_1'(\omega)=0,$$

$$F_1''(\omega)=-M_4\wp''(\omega)/(\epsilon_1-N_4)^2.$$

These values substituted in (16) give

$$(21)\quad H(t)+K(u)=\frac{M_4\wp''(\omega)}{2(\epsilon_1-N_4)^2}\left[\frac{1}{a_1-a_4+\frac{M_1}{\wp(2t-\omega)-N_1}-\frac{M_4}{\epsilon_1-N_4}}\right.$$

$$-\frac{1}{a_1 - a_4 + \dfrac{M_1}{\wp(2iu+\omega) - N_1} - \dfrac{M_4}{\epsilon_1 - N_4}}\Bigg].$$

For the reduction of (21) the following formulas are useful:

$$(22)\quad R(z) = A(z-a_1)(z-a_2)(z-a_3)(z-a_4),$$

$$R'(a_1) = A(a_1-a_2)(a_1-a_3)(a_1-a_4) = 4M_1,$$

$$R''(a_1) = 2A[(a_1-a_2)(a_1-a_3) + (a_1-a_3)(a_1-a_4) + (a_1-a_4)(a_1-a_2)] = 24N_1$$

[similarly for M_4 and N_4],

$$\epsilon_1 = \frac{A}{12}[(a_1-a_2)(a_3-a_4) + (a_1-a_3)(a_2-a_4)],$$

$$\epsilon_2 = \frac{A}{12}[(a_2-a_3)(a_1-a_4) + (a_2-a_1)(a_3-a_4)],$$

$$\epsilon_3 = \frac{A}{12}[(a_3-a_1)(a_2-a_4) + (a_3-a_2)(a_1-a_4)],$$

$$\epsilon_1 - N_4 = \frac{A}{4}(a_3-a_4)(a_4-a_2),$$

$$\epsilon_1 - N_1 = \frac{A}{4}(a_1-a_2)(a_3-a_1),$$

$$a_1 - a_4 - \frac{M_4}{\epsilon_1 - N_4} = 0.$$

With the aid of (17) also, the result for this case is

$$(23)\quad H(t) + K(u) = -(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)\left[\frac{1}{\wp(2t) - \epsilon_1} - \frac{1}{\wp(2iu) - \epsilon_1}\right].$$

It will be noticed that ϵ_2 and ϵ_3, which are conjugate, enter symmetrically, ϵ_1 being real. When all of the roots of $R(z) = 0$ are imaginary and accordingly all of the ϵ's are real, the choice of z_0 as in the preceding case reproduces (23).

The final forms for each case up to this point are the same as given by Haentzschel, though the method is different, in that one formula, (16), gives the resolved form for all cases.

For the last and most general case, where F is given by (8), the complete resolution into separate functions of t and u respectively would otherwise be obtainable only with much difficulty.

Although (16), the principal formula in this paper, is complete theoretically, there are three ways in which simplifications may be made in the general case. The first depends upon changes in the form of F which the writer has given in the *Archiv der Mathematik und Physik*, III, volume 23, Heft 1, and volume 24, Heft 4. The second involves certain simplifications in $F_1(\omega)$ which, with other related matters, are intended for a later paper, while the third consists of the following substitutions derived from the present discussion. In the last member of (16), s in F in the second denominator is replaced by $\wp(2t - \omega)$, also

$$(24) \qquad S = \frac{4(\epsilon_1 - \epsilon_2)^2(\epsilon_1 - \epsilon_3)^2[\wp(2t) - \epsilon_2][\wp(2t) - \epsilon_3]}{[\wp(2t) - \epsilon_1]^3}$$

and similarly for the first denominator.

In forming $F_1(\omega)$, z_0 is replaced by its conjugate, but s is simply ϵ_1 which is real, while S is zero.

$F_1''(\omega)$ may be computed from (6) thus depending on $F_1(\omega)$, i. e.,

$$(25) \qquad F_1''(\omega) = 2[AF_1^3(\omega) + 3BF_1^2(\omega) + 3CF_1(\omega) + B'].$$

As a summary, it may be stated that Wangerin made the solution of the original potential problem depend upon expressing a certain function of an unknown function as a sum of two functions $H(t)$ and $K(u)$, and obtained some simple values of the unknown function. Haentzschel applied a formula due to Weierstrass thus finding a general value of the unknown function, and he also computed the values of the two functions H and K for special cases only. Formulas (16) and (25) above give H and K explicitly in terms of the general function mentioned.

UNIVERSITY OF PENNSYLVANIA,
August 1, 1916.

NOTE CONCERNING THE NUMBER OF POSSIBLE INTERPRETATIONS OF ANY SYSTEM OF POSTULATES.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society December 27, 1917.)

In this note it is assumed that a mathematical system of postulates contains one or more undefined terms and that at least one of these denotes an element, i. e., a thing as distinguished from a relation. The assignment of an admissible meaning, or value, to each of the undefined elements of a postulate system will be spoken of as an interpretation of the system. By "admissible" meanings are meant meanings that satisfy the postulates or that, in other words, render them true propositions.

A postulate system may be such that from a given interpretation of it a second interpretation may be derived, from the second a third one, from the third a fourth, and so on, in such a way that mathematical induction is available for proving that the system admits of a denumerable infinitude of different interpretations. Such a system, for example, is Hilbert's system for euclidean metric geometry, where the undefined element-names are point, line, and plane. It is well known that one interpretation of this system results from allowing the term point to mean an ordered triad (x, y, z) of real numbers, the term plane to mean the class of triads satisfying an equation of the form $Ax + By + Cz + D = 0$ where not all the coefficients vanish, and the term line to mean the class of triads satisfying a pair of such equations.

If now we let the equations

$$(1) \quad \pi_1 + \lambda\pi_2 = 0, \qquad \pi_3 + \mu\pi_4 = 0, \qquad \pi_5 + \nu\pi_6 = 0$$

represent three pencils of planes, where the term plane has the meaning above assigned, it is evident that we may get a second interpretation of the Hilbert system by agreeing that the term point shall mean a triad (λ, μ, ν) of planes selected as indicated from the given pencils, that by a plane shall be meant a congruence, $a\lambda + b\mu + c\nu + d = 0$, of such triads,

and by a line the one-parameter system of triads common to two such congruences.

To obtain a third interpretation it is sufficient to regard the equations (1) as now representing three pencils of congruences of the foregoing variety and then to take for a point a triad of such congruences, for a plane a congruence of such triads of congruences, and for a line a one-parameter system of them as before.

The process exemplified in obtaining the third interpretation from the second obviously admits of endless repetition, yielding a denumerable infinitude of interpretations of which no two are identical.

The sequence of interpretations thus obtainable, far from including all possible interpretations of the system in question, excludes infinitely many possibilities. One of these is the familiar interpretation which results from agreeing that by a point shall be meant an ordinary point of ordinary inversion space a specified point O of which is treated as non-existent, that a plane shall signify a sphere that would contain O if O were not treated as non-existent, and that a line shall mean the (defective) circle common to two such defective spheres. Starting with this interpretation it would be easy to derive in the way above exemplified an endless sequence of interpretations differing from one another and from those of the foregoing sequence. It is not difficult to see that it would be possible to devise an endless variety of such sequences.

It is noteworthy that in case of any one of the interpretations belonging to either of the two sequences above indicated the ensemble of things denoted by the term plane and the ensemble denoted by the term line are subclasses of the ensemble denoted by the term point; and it is sometimes assumed that this relation of the three terms is required by the Hilbert postulates. That the relation is not required may be shown by an example as follows. In this interpretation the term point is to mean a triad (λ, μ, ν) of lines selected one from each of the pencils

$$(2) \quad L_1 + \lambda L_2 = 0, \qquad L_3 + \mu L_4 = 0, \qquad L_5 + \nu L_6 = 0,$$

which may be in a same plane or in three different planes; the term plane is to signify a congruence $a\lambda + b\mu + c\nu + d = 0$ of conics selected from the conic pencils

(3) $C_1 + \lambda C_2 = 0, \quad C_3 + \mu C_4 = 0, \quad C_5 + \nu C_6 = 0,$

taken to be in a same plane or in different planes; and the term line is to denote the one-parameter system of quadric surfaces common to a pair of congruences

(4) $\alpha\lambda + \beta\mu + \gamma\nu + \delta = 0, \quad \alpha'\lambda + \beta'\mu + \gamma'\nu + \delta' = 0$

of quadric surfaces selected from, say, three pencils of quadrics

(5) $Q_1 + \lambda Q_2 = 0, \quad Q_3 + \mu Q_4 = 0, \quad Q_5 + \nu Q_6 = 0.$

The questions arise: Does every postulate system admit of an infinite number of interpretations? If so, is the number the same for all systems? If it is, what is the number?

Let S be any postulate system. By hypothesis it contains at least one undefined element-name. Denote it by e. Let I be an interpretation of S. Suppose the meaning assigned in I to e is e'. Then all instances, $e_1', e_2', e_3', \cdots$, of e' satisfy the postulates of S. Let O be any object of thought whatever. This object and any given instance of e' together constitute a pair. Let ϵ be the class of such pairs. Evidently a one-one correspondence subsists between the instances of e' and those of ϵ (the members of the class ϵ). If two or more instances, $e_1', e_2', \cdots$, of e' satisfy some relation, we may say that the corresponding instances, $\epsilon_1, \epsilon_2, \cdots$, of ϵ also satisfy the same relation, provided we agree to say, as we evidently may, that any instances of ϵ shall be regarded as satisfying a given relation whenever the e'-instances contained in the ϵ-instances satisfy the relation. Accordingly, if we assign to e the meaning ϵ instead of the meaning e', we thereby give S a new interpretation I'. As many different interpretations may be thus obtained as there are objects O. In general the difference between two such interpretations will be trivial but it will never be nothing. The conclusion is that any postulate system admits of any given infinite number of interpretations.

COLUMBIA UNIVERSITY,
December, 1917.

DARBOUX'S PRINCIPLES OF GEOMETRY.

Principes de Géométrie Analytique. By GASTON DARBOUX. Paris, Gauthier-Villars, 1917.

A MONTH before his death Darboux wrote the preface to this, his last treatise. It presents in coordinated form lectures which he had delivered at intervals since 1872 either at the Sorbonne or the Ecole Normale. In the preface he announces that "the essential object of the book is the precise development of the notions relative to the imaginary and to infinity, and to show that in geometry they have the same place and importance as has been attributed to them in analysis for a long time." This idea is one which had very strong hold on Darboux's mind during his whole life. We find evidences of it in his first treatise, Sur une Classe remarquable de Courbes et de Surfaces algébriques et sur la Théorie des Imaginaires, published in 1872, which deals largely with material presented in better form in part of the present volume. He touched upon the idea in his address before the Fourth International Congress at Rome in 1908. Moreover, the results of his own researches, in which isotropic elements are so effectively used, bear fitting testimony to the force of his contention.

The treatise consists of five books, or parts. The first has an introductory chapter dealing briefly with the history of the development of the ideas of infinity and imaginary elements, and an indication of the value of these entities and of homogeneous coordinates. Darboux remarks that the moderns differ from the ancients in their aim to reduce geometrical problems to general principles. His own efforts along this line in the field of differential geometry were rewarded with results which constitute one of his chief glories. The next three chapters deal respectively with tetrahedral coordinates, homographic point transformations, anharmonic ratio, abridged notation of Bobillier and plane homologies. The treatment is essentially the same as is to be found in other developments of these ideas from the analytic point of view. It is necessary to treat these ideas fully, since the treatise is intended for students who have had merely an introductory course in cartesian analytical geometry of two dimensions. Similar characterization may be given the next three chapters,

which are short, and which deal with biaxial homologies, the principle of duality and correlation respectively. The last chapter of the first book deals with the well known projective properties of conics treated analytically.

In the second book isotropic elements are defined analytically and used in the development and refinement of the conceptions of distance and angle. The first chapter deals with metrical relations in the plane. The formula of Laguerre for the angle α between two lines D and D', namely, that $e^{2i\alpha}$ is equal to the cross ratio of the lines D, D' and the isotropic directions of the plane, is derived, and applied to the derivation of the formulas of the trigonometry of triangles, whether real or imaginary. The isotropic lines through two points A, B in a plane meet in two other finite points A', B' called by Darboux associate to the first. If M is any other point of the plane, we have

$$\frac{AM}{BM} = (-1)^k e^{\frac{1}{2}\widehat{B'MA'}},$$

k being an integer. By means of this formula Darboux shows that in a double infinity of ways the circle is the locus of a point whose distances from two fixed points, or poles, are in constant ratio. Generalizing this result, he studies during the remainder of the chapter the curves satisfying the condition $R_1R_2\cdots R_n = kr_1r_2\cdots r_p$, where k is constant, R_i is the distance of any point of the curve to a fixed point A_i and r_j the distance to a point a_j.

In chapter two the methods and ideas previously developed are applied to the determination and study of two classes of curves, obtained as solutions of the two problems:

To find the curve whose tangent at any point intercepts on n fixed lines of the plane segments which, measured from any fixed points on the lines, are in the most general linear relation.

To find the curve enveloped by a line which forms with n couples of fixed lines n triangles whose perimeters are in linear relation.

The imaginary circle at infinity, the isotropic plane, and null sphere, or isotropic cone, are defined analytically in chapter three; the idea of distance is made precise, and the angle between two lines is expressed in terms of cross ratio.

One of the best examples of the simplicity introduced into analytical geometry by the use of isotropic elements is fur-

nished by taking the rectilinear generators of the sphere for its parametric lines. These parameters are introduced in the fourth chapter, and the relation between rotations of the sphere and the linear fractional transformation is established. Having proved that the cross ratio of any four points on a circle of the sphere is equal to the cross ratio of the four generators of the one system through these points, and also of the four generators of the second system through the points, Darboux establishes the following fundamental formula

$$\frac{\tan \frac{1}{2}AM}{\tan \frac{1}{2}AM'} = e^{i\widehat{\gamma\gamma'}},$$

where AM and AM' are two arcs of great circles, and γ and γ' are poles of these circles, suitably chosen so that conventions as to signs shall be maintained. This formula forms the basis for the derivation, in the next chapter, of the formulas for the trigonometry of the sphere, such that the sides of the triangle are not required to be real, nor in the real case restricted in magnitude.

If A and B are two points of the sphere not on the same generator, the two generators through A meet the two generators through B in two points A' and B' other than points on the circle at infinity. The segments AB and $A'B'$, which evidently are polar to one another with respect to the sphere, are called associate segments. If a and b are points diametrically opposite to A and B, the four generators of the first system d, d_1, d_2, d_3 through these points meet the four generators of the second system δ, δ_1, δ_2, δ_3 in twelve points at finite distance. We indicate them as follows:

$$\begin{array}{cccccccccccc} A & B & A' & B' & A'' & B'' & a & b & a' & b' & a'' & b'' \\ d\delta_1 & d_2\delta_3 & d_2\delta_1 & d\delta_3 & d\delta_2 & d_1\delta_3 & d_1\delta & d_3\delta_2 & d_1\delta_2 & d_3\delta & d_2\delta & d_3\delta_1. \end{array}$$

In the last chapter of the second book, Darboux makes use of this enlarged system of points to study analytically the geometry on the sphere. He establishes the following fundamental system of formulas in which γ_A denotes the distance of any point M on the sphere to the point A, and S_{ABM} the area of the triangle ABM:

$$\frac{\gamma_A\gamma_B}{\gamma_{A'}\gamma_{B'}} = \frac{\gamma_a\gamma_b}{\gamma_{a'}\gamma_{b'}} = \cos\frac{AB}{2}, \quad e^{i\widehat{AMB}} = \frac{\gamma_{A'}\gamma_{b'}}{\gamma_{B'}\gamma_{a'}},$$

$$e^{i/2}S_{AMB} = \frac{\gamma_{b'}}{\gamma_{a'}}, \quad e^{i/2}S_{a'b'M} = \frac{\gamma_A}{\gamma_{B'}},$$

$$e^{i(\widehat{BAM}+\widehat{MBA})} = \frac{\gamma_{B'}\gamma_{b'}}{\gamma_{A'}\gamma_{a'}} = \frac{\sin B'M}{\sin A'M}.$$

These equations are applied to the determination and study of certain curves on the sphere analogous to those for the plane previously referred to.

Book Three is devoted to the proof of the celebrated theorems of Poncelet concerning polygons inscribed in one conic and circumscribed to a second conic. The analysis is rendered simple and elegant by the use of a coordinate system in the plane suggested by the Chasles representation of a quadric on a double plane. Each point of the plane is determined by the intersection of two tangents to a basal conic K. By a convenient choice of the equation of K the equations involved are given a simple form. After developing this coordinate system, the following theorems of consequence in the subsequent study are established:

A. If n lines are defined by equations of the form $A_i = 0$ $(i = 1, \cdots, n)$, the general equation of curves of order $n - 1$ passing through all the common points of the lines is

$$\frac{a_1}{A_1} + \cdots + \frac{a_n}{A_n} = 0,$$

the constants a_i being arbitrary.

B. If a curve of order $n - 1$ contains all the points of intersection of n tangents to a conic, it contains also the intersections of an infinity of other systems of n tangents to the same conic, each system being determined by a value of a certain parameter.

These theorems are applied at the beginning of the second chapter to the proof of Poncelet's theorem: When a polygon is circumscribed to a conic K and inscribed in a conic C, there exists a continuous suite of polygons of the same number of sides satisfying the same conditions. Moreover, in each case the points of intersection of any two sides of the polygon describe a conic or a straight line. After obtaining further properties of Poncelet polygons, Darboux gives an analytical explanation of the impossibility of obtaining such polygons when the conics C and K are chosen arbitrarily.

When a polygon of n sides is circumscribed to a conic K and $n-1$ of its vertices, in a displacement, describe conics in the same tangential pencil* with K, the other vertex describes a conic of the pencil. This general theorem of Poncelet is the subject of the last chapter of the third book. Darboux gives a geometrical proof of the case $n = 3$, and then extends the proof to the general case. Afterwards an analytical proof of the theorem is given. The chapter closes with a discussion of the forms given by Chasles to this theorem.

In his early writings Darboux made use of the ideas of distance and angle which grew out of Cayley's work. In 1895 he developed in a course of lectures at the Sorbonne the theory of Cayleyan geometry. In revised form these lectures constitute Book Four of the present treatise. In the first chapter we find an analysis of the origin of Cayleyan geometry. There is a full geometrical and analytical discussion of the absolute and of the consequences of its use.

Displacements in Cayleyan space are those homographic transformations which leave invariant the absolute, and thus preserve distances and angles. Darboux shows, in the second chapter, that in order to obtain a continuous transformation of this kind it is necessary and sufficient to subject the parameters of the two families of rectilinear generators of the absolute to any two linear substitutions.

When one of these substitutions is the identity, in the corresponding displacement all points of space are displaced the same distance. For the general case, in every displacement there are two lines, D and Δ, polar with respect to the absolute, which slide over themselves; they are called the axes of the displacement and are real for a real displacement. These displacements are of three types, all of which may be real: helicoidal, when four points only of the absolute remain fixed; rotations, when all the points of either axis remain fixed; half-turns (retournements), when all the points of both axes are fixed. These results are derived geometrically, and then the equations of the displacements of various kinds are obtained in simple form, when the equation of the absolute is $xy + zt = 0$.

In the first part of chapter three the formulas of Cayleyan trigonometry are derived. Then follow expressions for volume and area. A sphere being defined as the locus of points equi-

* All the conics tangent to four lines form a tangential pencil.

distant from a fixed point, two cases present themselves according as the latter point does or does not lie on the absolute.

The greater part of this book deals with finite elements, but occasionally there are sections dealing with differential geometry. The exposition throughout is so clear and simple that a college student might very well use this treatise as an introduction to this field of study.

The last book deals with inversion and with geometric concepts in whose field this transformation plays an important rôle. As a student in Paris, Darboux became interested in this subject and his first papers dealt with curves and surfaces for which these transformations are of peculiar significance. Out of them grew his remarkable thesis, Sur les surfaces orthogonales, and his first treatise previously referred to. In these publications are to be found the essential elements of the present book, so that here we have Darboux renewing his youth and embellishing its products with the style and finish acquired through the years.

Darboux begins with the following geometrical definition of inversion: Consider a sphere S, called the principal sphere of the inversion; the transform of a point M is the unique point M' such that the null spheres with centers at M and M' meet S in the same circle. From this he derives the well known equations of inversion. The first chapter is devoted to the determination of the essential properties of inversion. It should be noted that these are obtained by geometrical reasoning and not by analytical processes. In particular, isotropic lines are transformed into isotropic lines. Here is introduced the isotropic developable, that is, the developable whose generators are isotropic lines. These surfaces, which are transformed into surfaces of the same kind by inversion, play an important rôle in many of Darboux's writings. In one of his first papers he called attention to the fact that an isotropic developable circumscribing a surface meets it along a line of curvature. A double curve on the isotropic developable circumscribing a surface is the locus of null spheres doubly tangent to the surface. These are called the focal curves of the surface, and each point is a focus of the surface. This definition is a generalization of Plücker's for foci of a plane curve and is in accord with Chasles's definition of focals for quadrics.

If we have five fixed spheres such that any two meet one another orthogonally, and s_i denotes the power of a point with respect to the sphere S_i and r_i its radius, the following identities hold for any point:

$$(1) \qquad \sum_{i=1}^{5}\left(\frac{s_i}{r_i}\right)^2 = 0, \quad \sum_{i=1}^{5}\frac{s_i}{r_i^2} = -2, \quad \sum_{i=1}^{5}\frac{1}{r_i^2} = 0.$$

When one of the spheres S_i is a plane, these identities hold provided that in the first the term s_i/r_i be replaced by twice the distance from the point to the plane, and in the last two identities the corresponding term be omitted. The quantities x_i defined by

$$x_i = \lambda\frac{s_i}{r_i},$$

where λ is a factor of proportionality, are called the pentaspherical coordinates of the point. From the first of (1) we see that they must satisfy the condition

$$(2) \qquad \sum_{i=1}^{5} x_i^2 = 0.$$

Moreover, the factor λ is determined in accordance with the second of (1) by

$$\sum\frac{x_i}{r_i} = -2\lambda.$$

This coordinate system is introduced in the second chapter and various properties are obtained, including the relations between the pentaspherical and rectangular coordinates of a point. In Darboux's opinion these coordinates bear a peculiar relation to inversion just as homogeneous point coordinates do to homographic transformations and tangential coordinates to correlations. The contents of this chapter are in the main equivalent to those set forth by Darboux in the first volume of his Leçons sur la Théorie générale des Surfaces with the difference that in the present treatise there are fewer applications to differential geometry. In the present treatment these coordinates appear in their historical setting. For Darboux was lead to invent this system as a result of his study of cyclides, i. e., the surfaces of the fourth degree with the circle at infinity as a double curve and the surfaces of the third degree passing through this circle. The remainder of the treatise deals with the study of these surfaces.

According to Moutard, a surface is anallagmatic when it is transformable into itself by inversion. He showed that the envelope of the spheres orthogonal to a fixed sphere S possesses this property when S is the principal sphere of the inversion and that all anallagmatic surfaces can be obtained in this way. He called the center of S the principal pole of the surface and the locus of the centers of the spheres the directrix surface. He announced the following properties of the cyclides:

Every cyclide is anallagmatic in five different ways; the line joining any two principal poles is perpendicular to the plane of the other three; any two of the five principal spheres cut orthogonally; the directrix surfaces are homofocal quadrics, which are central quadrics when the surface is of the fourth order and paraboloids when of the third order; each sphere doubly tangent to the surface meets it in two circles; each principal pole of a cyclide of the fourth order is the vertex of a quadric cone whose generators are doubly tangent to the surface, the curve of contact being on a sphere concentric with the directrix quadrics, and through each point of the surface pass in general ten circular sections.

These results are established by Darboux in the third chapter by analytical processes. The equation in cartesian coordinates of the general cyclide is of the form

$$\varphi_0(x^2 + y^2 + z^2)^2 + 2\varphi_1(x^2 + y^2 + z^2) + \varphi_2 = 0,$$

where φ_0, φ_1, φ_2 denote functions of x, y and z of the degrees indicated by the subscripts. Darboux gives these functions normal forms before proceeding to the proof of the above result. He establishes also the existence of quadrics V inscribed in the cyclide of fourth order along curves lying on concentric spheres; the five cones referred to above are quadrics. Since a cyclide of the third order can be regarded in an infinity of ways as a homographic transform of a general cubic surface, certain properties of the latter are studied in relation to the generalized inversion proposed by Hirst.

We remarked that Darboux was led to pentaspherical coordinates by his study of the cyclides. When the five fixed spheres referred to in Moutard's results are taken for the coordinate spheres of a pentaspherical system, the equation of the general cyclide is of the form

$$\sum_{i=1}^{5} a_i x_i = 0,$$

as Darboux shows in chapter four. In this chapter and the next he derives from this analytical point of view the properties of the cyclides mentioned above, and many others.

Darboux's first important discovery was the existence of triply orthogonal systems of homofocal cyclides which constitute the most direct generalization of the well known triple system of confocal quadrics. The equation of such a system in pentaspherical coordinates is

$$\sum_{i=1}^{5} \frac{x_i^2}{a_i - \lambda} = 0,$$

where the a's are constants, and λ is a parameter, such that through each point of space pass three cyclides corresponding to three values of λ. Chapter six is devoted to a study of these systems.

The final chapter of the treatise deals with a point transformation first made known by the author in 1865. This transformation is defined analytically by the equations

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{2a}{x^2 + y^2 + z^2 + a},$$

where a is a positive or negative constant. A combination of this transformation and the general homographic one gives the most general transformation which makes a plane correspond to a sphere. To every point $M(X, Y, Z)$ there correspond two points m and m' of coordinates x, y, z and x', y', z', which are the centers of null spheres meeting the sphere S, defined by

$$x^2 + y^2 + z^2 - a = 0,$$

in the circle of intersection of S and the polar plane of M with respect to S. We will not enumerate the various results obtained by Darboux, but will content ourselves with his observation that this transformation permits one to substitute for Cayleyan geometry a new geometry in which circles orthogonal to a sphere play the rôle of straight lines.

The reader of this treatise is sure to be impressed by the striking use which Darboux has made of isotropic elements. By the use of them and of frequent geometrical reasoning he has been enabled to avoid tedious stretches of analytical work. It is the simplicity and elegance of the processes which impress one.

It seems almost superfluous to refer to the style of presentation of the results in this treatise. Every one in the least familiar with Darboux's writings is conscious of their lucidity and charm of style. And here we have the final product of this great expositor.

L. P. EISENHART.

SHORTER NOTICES.

On Growth and Form. By D'ARCY W. THOMPSON. Cambridge, University Press, 1917. 8vo. 16+793 pages. Price 21 shillings.

UNTIL his recent appointment to the chair of natural history at the University of St. Andrews, the author of this work was professor of natural history at University College, Dundee. He has long been prominent as a member of various fishery boards and conferences, serving, for example, as the British delegate to the Bering Sea fishery conference of 1897. He is the author of "A Glossary of Greek Birds," has edited Aristotle's Historia Animalium and various blue-books on North Sea investigations, and he has translated and edited Hermann Müller's Fertility of Flowers.

And now we have this elaborate volume containing the results of many years of observation and study in widely separated fields and constituting an "easy introduction to the study of organic form." The author tells us: "It is not the biologist with an inkling of mathematics, but the skilled and learned mathematician who must ultimately deal with such problems as are merely sketched and adumbrated here. I pretend to no mathematical skill, but I have made what use I could of what tools I had; I have dealt with simple cases, and the mathematical methods which I have introduced are of the easiest and simplest kind."

After the Introduction we have chapters with the following titles: 2. On magnitude; 3. The rate of growth (106 pages); 4. On the internal form and structure of the cell; 5. The forms of cells; 6. A note on absorption; 7–8. The form of tissues or cell-aggregates; 9. On concretions, spicules, and spicular skeletons; 10. A parenthetic note on geodetics; 11. The

logarithmic spiral* (94 pages); 12. The spiral shells of the foraminifera; 13. The shapes of horns, and of teeth or tusks: with a note on torsion; 14. On leaf-arrangement or phyllotaxis; 15. On the shapes of eggs and of certain other hollow structures; 16. On form and mechanical efficiency; and 17. On the theory of transformations or the comparison of related forms.

The whole work is written with fluency, lightness of touch, and notably attractive phrasing fraught with a wealth of suggestion. Many historical notes and full bibliographies indicate the author's thorough familiarity with the literature of his subject. The mathematics involved is indeed elementary, although some of the discussion refers to results whose derivation must have required the use of the theory of least squares and of probabilities† (for example, in Chapter 3), while other parts involve such subjects as are summarized by O. Fischer in his Physiologische Mechanik‡ (for example in Chapters 2 and 16). In conclusion let us note a few of the vast number of special topics discussed in a most interesting manner.

Some of the illustrations of "hexagonal symmetry" in the seventh chapter are: (1) artificial cellular tissue formed by the diffusion in gelatine of drops of a solution of potassium ferrocyanide; (2) "hexagonal epithèlium-cells of the pigment layer of the eye, external to the retina"; (3) optical sections of soap froth imprisoned between glass plates; and (4) the bee's cell. We are told that "the curious reader may consult Sir Thomas Browne's quaint and beautiful account in the Garden of Cyrus,§ of hexagonal (and also of quincuncial) symmetry in plants and animals which 'doth neatly declare how nature Geometrizeth, and observeth order in all things.'" In the seven-page account of the bee's cell Thompson sketches the

* In connection with this chapter it is noted that "a great number of spiral forms, both organic and artificial, are described and beautifully illustrated in Sir T. A. Cook's Curves of Life, 1914, and Spirals in Nature and Art, 1903."

† An excellent American work which gives a good idea of some kinds of mathematics employed in biology is C. B. Davenport's Statistical Methods with Special Reference to Biological Variation. Third edition. New York, 1914.

‡ Encyklopädie der mathematischen Wissenschaft, Band IV, 1, II, Heft 1, 1904. Not referred to by Thompson.

§ Published in 1658.

correct history* of discussions of its form (by Pappus, Maraldi, Réaumur, Koenig, Maclaurin, Lhuilier, Buffon, P. Huber and Darwin), and the derivation, by calculus, of its minimum property.

The mathematician will more than once take exception to statements made in the two paragraphs on pages 440–441 concerning "geodetic" curves on surfaces. These curves are introduced in connection with the discussion of spicules, helicoid formations as in the coil which stiffens the tracheal tubes of an insect (Chapter 10), etc.

Discussion of leaf-arrangement leads to the series of ratios: 2/3, 3/5, 5/8, 8/13, 13/21, 21/34, 34/55, 55/89, 89/144, which are simply convergents of the continued fraction $\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$, etc. Thompson refers, incorrectly, to this "celebrated series" as "commonly called the Fibonacci series." This designation would be more correctly applied to the series of numbers 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 (each term of which, after the second, is the sum of the two preceding) which comes up in the solution of a rabbit problem discussed by Leonardo of Pisa in his Liber Abbaci.† Fibonacci's series is sometimes called Lamé's series.‡ It was well known to Kepler, who discussed it in connection with flower forms,§ and it is closely connected with the division into golden section‖ and with a number of mathematical investigations by Binet, Lamé, Lucas, Catalan, Schlegel, and Landau.¶

* Incorrect statements, from the time of Réaumur, have been made so often in this connection (e. g., Mach, Science of Mechanics, 1902, p. 453) that the corrections indicated by R. L. Ellis in his essay "On the Form of Bees' Cells" should be noted (Mathematical and Other Writings, Cambridge, 1863, p. 353).

† Scritti, tome 1, pp. 283–284.

‡ Thompson makes a slip in connecting this series with "Father Bernard Lami, a contemporary of Newton."

§ F. Ludwig, "Kepler über das Vorkommen der Fibonaccireihe im Pflanzenreich," *Bot. Centralbl.*, 1896, Bd. 68, p. 7.

‖ Two books on this subject are: (1) A. Zeising, Neue Lehren von den Proportionen des menschlichen Körpers aus einem bisher unbekannt gebliebenen und die ganze Natur und Kunst durchdringenden Grundgesetze. Leipzig, 1854; (2) F. X. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur und Kunst. Augsburg, 1885 —but Thompson regards the numbers of golden section as "devoid of biological significance" (p. 649).

¶ Cf. *Amer. Math. Monthly*, May, 1918.

In *L'Intermédiaire des Mathématiciens* for 1895* Monsieur Henri Brocard suggested a study of the "fruit de l'hélianthe" —a comparison of the arrangement of the florets of the sunflower with such curves as spirals, epicycloids and the involute of a circle. In 1909 a remarkable sunflower photograph taken by M. Brocard was published along with notes: (1) by J. W. N. Le Heux, who found "lignes rapellant tout à fait la spirale d'Archimède (ou la spirale logarithmique), ou même d'autres spirales, selon la loi de variation du rayon vecteur," and (2) by E. Maillet who wrote as follows:

"Je ne puis m'empêcher d'être frappé de l'analogie de disposition que présentent les spirales de la photographie de M. Brocard avec celles qu'a rencontrées M. le commandant Hartmann, dans ses études sur la déformation des métaux, pour le cas d'un cylindre creux soumis à une pression intérieure, ou extérieure, ou pour le cas d'une plaque poinçonnée en son centre [*Revue d'Artillerie*, novembre (page 99–100) et décembre 1894, 1895 et 1896].

"M. Mesnager a montré (*Revue d'Artillerie*, mars 1898) qu'on pouvait, au moins en partie, expliquer la production de ces lignes en admettant que, dans la déformation d'un corp donné soumis à des efforts: 1° le glissement tend à se produire en un point le long de l'élément plan suivant lequel la valeur absolue du rapport R de l'effort tangentiel à l'effort normal (cohésion comprise) est maximum; 2° ce glissement, qui constitue *une déformation permanente*, se produit effectivement dès que R atteint une valeur constante f (analogue à un coefficient de frottement) caractéristique du métal.

"Voir encore Appell, *Bull. Soc. Math.*, tome XXVIII, 1900, p. 66.

"Je ne suis pas assez botaniste pour dire si des phénomènes de déformation tout à fait analogues au point de vue de la Mécanique peuvent se produire dans le développement de la

* Tome 2, p. 134, question 524; reproposed 1902, tome 9, pp. 138–139. Notes: 1904, tome 11, p. 95; 1908, tome 15, pp. 173–174; 1909, tome 16, pp. 101–102.

In response to another question in *L'Intermédiaire* (1688, by Espanet, Dec. 1899, tome 6, p. 269), concerning surfaces of sea shells, more particularly those of spiral form, Brocard, Godefroy and Escott gave a number of references to the literature of the subject (1900, tome 7, p. 40; 1901, tome 8, pp. 167, 314; 1910, tome 17, p. 155), nearly all of which, and many more, seem to have been considered by Thompson. One notable work which he appears to have overlooked is Haton de la Goupillière's "Surfaces Nautiloïdes" which occupies almost the whole of the third volume of *Annaes scientificos da Academia polytechnica do Porto*, Coimbra, 1908.

fleur et du fruit de l'hélianthe (ou d'autres plantes) et si cela peut influer sur la disposition de la mosaïque ou même la déterminer."

According to the volume before us* it seems that there is no doubt about the logarithmic spiral arrangement in the sunflower, or indeed that we have systems intersecting isogonally. Reference is made to a careful and elaborate study by H. H. Church,† which has been adopted by T. A. Cook in his Curves of Life. But of Thompson's comment in this connection I shall not do more than quote a single sentence: "On the analogy of the hydrodynamic lines of force in certain vortex movements, and of similar lines of force in certain magnetic phenomena, Mr. Church proceeds to argue that the energies of life follow lines comparable to those of electric energy, and that the logarithmic spirals of the sunflower are, so to speak, lines of equipotential."

Professor Thompson's work is recommended as one of the most notable and most readable of scientific books appearing in the past decade.

R. C. ARCHIBALD.

Hvorledes Mathematiken I Tiden Fra Platon Til Euklid Blev Rationel Videnskab. By H. G. ZEUTHEN. Reprinted from the D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem. Afd., 8. Række, I. 5. Copenhagen, 1917. 183 pp.

ANY work by Professor Zeuthen on the history of mathematics, even though it be a reprint from the memoirs of an academy, deserves to be brought to the attention of other scholars than those who may chance to see the original publication. This is especially true when the memoirs of the academy are printed in a language not generally familiar to scholars and therefore are not as frequently consulted as those which appear in languages more nearly international.

In this particular case there is the more reason for calling attention to the memoir because a summary is given in the French language so that all scholars may have easy access to the argument and, with this as a guide and with a knowledge of German, may follow the more important details in the text itself.

* Pp. 639–640.

† Relation of Phyllotaxis to Mechanical Laws, Oxford, 1901–1903; cf., *Ann. of Botany*, vol. 15 (1901), p. 481.

It is only a commonplace truism that the mission of the historian is quite as much the interpretation of facts as the mere writing of chronicles. It is not the facts of the present war but the interpretation of these facts and the relation of their influence upon the world that will tax the powers of the future historian. It is this feeling that will strike the reader of Professor Zeuthen's memoir. What has been done is not to call attention to newly discovered materials but to study again the significance of certain important contributions to Greek mathematics in one of the most remarkable periods of the early growth of the science. Probably only a few years intervened between the death of Plato and the birth of Euclid; certainly less than half a century elapsed from the period of the former's activity to that of the rise of the Alexandrian school; and yet it is in this period that Professor Zeuthen finds the recognition of mathematics as a logical science.

Chapter I considers the period in general, characterized as it is by the transition from an intuitive geometry to the demonstrative form in which it has come down to us.

Chapter II discusses those features which characterize a "science raisonnée" such as Euclid sought to include in his Elements. These features include the definition, the postulate, and those "common notions" which we call the axioms. The importance of all of these was fully understood by Plato, and the influence of this master inspired his pupils to undertake those investigations that finally took scientific form in the work of Euclid.

Chapter III sets forth the demands made by Plato upon mathematics in its quality of a "science raisonnée." The master seems not to have been particularly dissatisfied with the progress of plane geometry, possibly under his own influence; but he felt that stereometry was not yet upon a scientific basis.

Chapter IV discusses the rise of the analytic method. The positive facts of elementary geometry were already fairly well known, but it was apparently Plato who suggested the analytic method, with the inverse synthetic form of proof, and it was the development of this method that made possible a work like Euclid's.

Chapter V records the names of some of those who appreciated the reform instituted by Plato and carried it on to fruition. These names include not only those whose major

interest was mathematics, but, it need hardly be said, the name of Aristotle whose contributions to the study of logic assisted in the efforts of the geometer to attain the ideals which Plato had set forth.

Chapter VI returns to the conception of geometry before the reform of Plato, or rather, to the pre-Platonic psychology of form. The problem is to set forth the primitive and intuitive images of people, the early apperception of similarity, of the relation of one form to another, and of the possibility of the displacement of figures without alteration.

Chapter VII enlarges upon the study of displacements as mentioned in the preceding chapter; for example, as shown in the earlier work of Pythagoras and in the much later work of the Indian scholar Bhāskara.

Chapter VIII brings the theory of elementary displacements to a climax in the Elements.

Chapter IX summarizes the growth of the idea of similarity, tracing it to its culmination in Euclid's work.

Chapter X is devoted to a consideration of the notion of angle. Professor Zeuthen shows that the general idea of angle, as distinct from the idea of perpendicularity, arose from the study of similar figures, which of course included congruence as a special case. The rise and use of the horn-shaped angle (keratoeides) is also mentioned.

Chapter XI considers the generalization of demonstrations, including the nature of incommensurables and the use of the idea of the infinitesimal. The influence of Zeno and Eudoxus upon Euclid's treatment of the incommensurable case and upon the work of Archimedes is shown. The influence of Aristotle upon Eudoxus and Euclid is compared with that of Dedekind upon Weierstrass.

Chapter XII relates to the generalization of propositions by Euclid to meet the demands imposed by Aristotle, and shows that his postponement of the geometric treatment of the quadratic was due to the desire to give the most general form possible.

Chapter XIII deals with the question of the ideality of geometric figures, a question by no means new when Plato called it to the attention of his pupils.

Chapter XIV discusses Euclid's treatment of stereometry, particularly with reference to the distinction between congruence and symmetry in a space of three dimensions.

Chapters XV and XVI treat of the Elements, the appreciation accorded to it as a scientific work during two thousand years, and its standing as a scientific work to-day. The influence of Euclid upon the work of Apollonius is set forth in terms of appreciation that it would be well for all teachers of mathematics to consider.

The work is of the scholarly character that one would expect from a man of Professor Zeuthen's exceptional attainments, and it is earnestly to be hoped that he will publish the entire memoir in the French language. Indeed, it should not be too much to hope that Professor Zeuthen may revise all of his historical works and publish them in a uniform French edition before he is compelled to relinquish his writing and to take the scholarly ease that he has so well earned.

DAVID EUGENE SMITH.

A Manual of Mathematics. By R. G. HUDSON and J. LIPKA. New York, Wiley, 1917. 132 pp. Price $1.00.

A Table of Integrals. By R. G. HUDSON and J. LIPKA. New York, Wiley, 1917. 25 pp. Price 15 cents.

THIS manual is taken from the mathematical section of the authors' manual for engineers. It contains the formulas of algebra, trigonometry, mensuration, plane and solid analytic geometry, differential and integral calculus, ordinary and partial differential equations, complex quantities and vectors. In addition to the usual tables—four-place—there is a table of conversion factors for the various scientific units. The formulas are systematically arranged and the section headings are printed in heavy type so that it is easy to locate any formula, a very desirable feature in such a book. The table of integrals is taken from the manual. The integrals are expressed in terms of the coefficients used in the integrand. While this makes the table more difficult for the printer than if substitutions and abbreviations had been made, it is much more convenient for the student.

THOS. E. MASON.

NOTES.

THE April number (volume 40, number 2) of the *American Journal of Mathematics* contains the following papers: "On the representation of functions in series of the form $\Sigma c_n g(x+n)$," by R. D. CARMICHAEL; "Transformations of planar nets," by L. P. EISENHART; "Orthogonal function sets arising from integral equations," by O. D. KELLOGG; "Complete systems of concomitants of the three-point and the four-point in elementary geometry," by C. H. RAWLINS, JR.; "Systems of pencils of lines in ordinary space," by A. L. MILLER; "Some contributions to the geometry of plane transformations," by TOBIAS DANTZIG; "Properties of a certain projectively defined two-parameter family of curves on a general surface," by PAULINE SPERRY.

AN announcement has just been issued that the *Bollettino di Bibliografia e Storia delle Scienze Mathematiche*, edited by Professor GINO LORIA, of Genoa, is about to begin a new series. This valuable journal will hereafter be published by the Libreria Scientifica D. Capozzi, Palermo, Via Croce dei Vespri, 8. The *Bollettino* has proved so useful to students of the bibliography and history of mathematics that the inauguration of a new series becomes a matter of interest to all mathematicians.

AT the meeting of the Edinburgh mathematical society held February 8, the following papers were read: By G. SMEAL, "The parameter method of iteration"; by Miss E. PAIRMAN, "A new form of the remainder in Newton's interpolation formula."

At the meeting of March 8, the following papers were read: By C. G. KNOTT, "Note on the quaternion theory of confocals"; by R. F. MUIRHEAD, (a) "A graphic method of determining the maximum bending moment for a rolling load on a uniform beam"; (b) "Geometrical notes on the accelerations of points of a rigid plane body moving in its own plane."

AT the meeting of the London mathematical society held February 14 the following papers were read: By A. C. DIXON,

"Note on functional equations which are limiting forms of integral equations"; by D. M. Y. SOMMERVILLE, "The singularities of trochoidal curves"; by L. J. MORDELL, "A statement of Fermat."

AT the meeting of the National academy of sciences at Washington, April 22–24, Professor EDWARD KASNER presented a paper on "Conformal geometry." Professor G. D. BIRKHOFF was elected a member of the academy.

GINN and Company announce that the second volume of Veblen and Young's Projective geometry is now in press.

THE Cambridge University Press announces the early publication of a Theory of functions of a complex variable, by A. R. FORSYTH. Longmans, Green and Company, London, are publishing a Differential equations by H. BATEMAN.

ON pages 183–267 of the Publications of the members of the University of Chicago 1902–1916, recently issued by the University of Chicago Press, are listed published doctors' theses and the publications of members of the staffs in the departments of mathematics, astronomy and astrophysics, and physics.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 8–August 16).—By Professor JAMES MACLAY: Fundamental mathematical concepts, five hours; Higher algebra, five hours.—By Professor EDWARD KASNER: Modern geometry, five hours; Continuous groups, five hours.—By Professor W. B. FITE: Theory of functions of a real variable, five hours; Differential equations, five hours.

COLUMBIA UNIVERSITY (academic year 1918–1919).—By Professor T. S. FISKE: Theory of functions, four hours.—By Professor F. N. COLE: Algebra, four hours.—By Professor JAMES MACLAY: Differential geometry of surfaces, four hours, first half-year; Applications of the elliptic functions, three hours, second half-year.—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathe-

matics, four hours.—By Professor C. J. KEYSER: Philosophy of mathematics, four hours.—By Professor EDWARD KASNER: Ballistics, two hours, second half-year; Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. C. A. FISCHER: Calculus of variations, three hours, second half-year.

The following university and college teachers of mathematics have recently entered the national military service:

Dr. W. L. CRUM, of Yale University, has joined the national army. Professor G. C. EVANS, of Rice Institute, has joined the signal corps and has gone to France. Professor W. E. MILNE, of Bowdoin College, has been made first lieutenant in the ordnance reserve corps. Dr. H. C. M. MORSE, of Harvard University, has joined the ambulance corps and is in France. Professor F. R. MOULTON, of the University of Chicago, has been commissioned major in the ordnance reserve corps. Professor A. H. NORTON, of Elmira College, and Professor L. P. SICELOFF, of Columbia University, are engaged in Y. M. C. A. work in France. Dr. J. F. RITT, of Columbia University, has joined the ordnance at the Aberdeen proving ground, Aberdeen, Md.

PROFESSOR E. W. BROWN, of Yale University, has been elected a member of the council of the American philosophical society.

AT Harvard University assistant professor J. L. COOLIDGE has been promoted to a full professorship of mathematics.

AT Wellesley College, Professor ROXANA H. VIVIAN, of the department of mathematics, has been made director of the department of hygiene and physical education. Professor CLARA E. SMITH is to exchange with Professor FLORENCE P. LEWIS, of Goucher College, for the coming academic year. Dr. MARY F. CURTIS, of Western Reserve University, has been appointed instructor in mathematics.

DR. W. G. SIMON has been appointed instructor in mathematics at Adelbert College, Western Reserve University.

PROFESSOR A. BENTELI, of the University of Bern, died November 10, 1917, at the age of seventy years.

PROFESSOR E. OTT, of the University of Bern, died November 17, 1917, at the age of seventy years.

PROFESSOR F. R. HELMERT, of the University of Berlin, died June 15, 1917, at the age of seventy-three years.

THE death is reported of Professor CYPARISSOS STÉPHANOS, of the University of Athens.

LEWIS B. CARLL, the blind author of the first American treatise on the calculus of variations (published in 1881), died March 22, 1918, at the age of seventy-four years.

PROFESSOR H. R. HIGLEY, of the Stevens Institute of Technology, died May 1, 1918, at the age of fifty-four years.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, list number 91, 149 titles in mathematics and physics. —Henry Sotheran and Co., 140 Strand, London, catalogue 770, exact and applied science, 928 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ASKWITH (E. H.). A course of pure geometry containing a complete geometrical treatment of the properties of the conic sections. New edition revised and enlarged. Cambridge, University Press, 1917. Demy 8vo. 11 + 284 pp. 7s. 6d.

CAREY (F. S.). Infinitesimal calculus. Section 2. London, Longmans, Green and Company, 1917. 10s. 6d.

CISOTTI (U.). Lezioni di analisi matematica, svolte nel r. Istituto tecnico superiore di Milano. Pavia, tip. Cooperativa, 1918. 8vo. 16 + 674 pp. L. 30.00

CRAWLEY (E. S.) and EVANS (H. B.). Analytic geometry. Philadelphia, E. S. Crawley, 1918. 8vo. 14 + 239 pp.

EVANS (H. B.). See CRAWLEY (E. S.).

INTERNATIONAL catalogue of scientific literature. Fourteenth annual issue. A: Mathematics. London, Harrison, 1917. 8vo. 8 + 163 pp. Paper. 15s.

MICHEL (C.). Cours d'algèbre et d'analyse. Paris, Alcan, 1916. 8vo. 10 + 860 pp.

RUSSELL (B.). Mysticism and logic, and other essays. New York, Longmans, Green and Company, 1918. 8vo. 8 + 234 pp. $2.50

SHAW (J. B.). Lectures on the philosophy of mathematics. Chicago and London, Open Court, 1918. 8 + 206 pp. $1.50

STADLER (G.). Etudes sur l'équation $\delta^p u/\delta z^p = a\Delta^m u = 0$. (Diss., Lund.) Lund, 1916. 4to. 6 + 89 pp.

II. ELEMENTARY MATHEMATICS.

BRENKE (W. C.). Advanced algebra. New York, Century Company, 1917. 8vo. 8 + 196 pp. Cloth. $1.25

——. Elements of trigonometry with tables. New York, Century Company, 1917. 8vo. 6 + 122 + 40 pp. Cloth. $1.25

LYNDE (L. E.). See NEWTON (F. E.).

NEWTON (F. E.) and LYNDE (L. E.). Exercises in plane geometry with syllabus and definitions. Designed for the second reading of the subject. 3d edition. Andover, Mass., Andover Press, 1918. 86 pp. $0.75

PAGLIERO (G.). Come si usano i logaritmi per calcolare rapidamente: teoria e pratica; tavola dei logaritmi a quattro decimali. Torino, ditta G. B. Paravia, 1917. 8vo. 47 pp. L. 2.00

THOMPSON (J. B.). The art of teaching arithmetic. New York, Longmans, Green and Company, 1917. 8vo. 8 + 295 pp. $1.35

UNWIN (W. C.). Short logarithmic and other tables. 6th edition. London, Spon, 1918. 1s. 6d.

ZORETTI (L.). Tables numériques usuelles à l'usage des ingénieurs, des étudiants des facultés, des élèves des lycées et des écoles primaires supèrieures, etc. Paris, Gauthier-Villars, 1917. 52 pp. Cartonné. Fr. 3.00

III. APPLIED MATHEMATICS.

AMERICAN ephemeris and nautical almanac, 1919. Washington, Government Printing Office, 1917. 8vo.

ATTWOOD (E. L.). A text-book of laying off, or the geometry of shipbuilding. 2d edition. New York, Longmans, Green and Company, 1918. 8vo. 8 + 123 pp. $2.00

BUCHERER (E.). Grundzüge der mathematischen Geographie. Basel, Krebs, 1917. 8vo. 40 pp.

DANFORD (R. M.). See MORETTI (O.).

EFFEMERIDI astronomiche ad uso dei naviganti per l'anno 1918 (Istituto idrografico della r. marina). Genova, tip. Istituto idrografico della r. marina, 1917. 8vo. 6 + 191 pp. L. 2.00

FLAMARD (E.). Calcul des systèmes élastiques de la construction. (Encyclopédie industrielle.) Paris, Gauthier-Villars, 1918. 8vo. 6 + 200 pp. Fr. 12.00

FLAMMARION (C.). Annuaire astronomique et météorologique pour 1918. Paris, Flammarion, 1918. 364 pp. Fr. 3.00

JAEGER (F. M.). Lectures on the principle of symmetry and its applications in all natural sciences. Amsterdam, Uitgevers Maatschappy Elsevier, 1917. 8vo. 333 pp.

MEMENTO Oppermann, à l'usage des ingénieurs, architectes, agents voyers, conducteurs de travaux, mécaniciens, industriels, entrepreneurs. Paris, Béranger, 1918. 16mo. 268 pp. Fr. 6.00

MORETTI (O.). Notes on training; field artillery details. Prepared under the direction of R. M. Danford. New Haven, Yale University Press, 1917. 8vo. 12 + 275 pp.

RODET (J.). Notions d'acoustique. Les instruments de musique. Le télharmonium. Paris, Gauthier-Villars, 1917. 8vo. 4 + 96 pp. Fr. 3.50

STOHLER (H.). Mathematische Geographie und sphärische Trigonometrie. Basel, Basler Drug und Verlags-Anstalt, 1916. 8vo. 96 pp.

TSCHAPPAT (W. H.). Text-book of ordnance and gunnery. New York, Wiley, 1917. 8vo. 10 + 705 pp. Cloth. $6.50

VALBREUZE (R. DE). Notions sommaires d'électrotechnique. Paris, chez l'auteur, 12, rue Pelleport, 1918. 8vo. 180 pp. Fr. 6.00